Efficient Monte Carlo for Gaussian Fields and Processes

Jose Blanchet (with R. Adler, J. C. Liu, and C. Li)

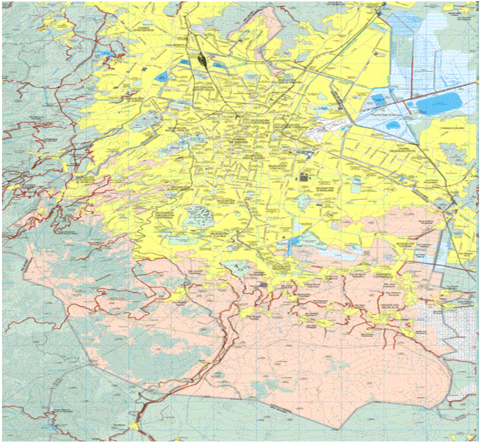
Columbia University

Nov, 2010

Outline

A Motivating Application

- Contamination level in a geographic area... (yellow area = Mexico City)



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- Adler, Blanchet and Liu (2010): **Algorithm for conditional sampling with ε relative precision in time**

$$\mathbf{Poly}\{\varepsilon^{-1}\log[1/p(\textbf{high excursion})]\}$$

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- Another motivating application: Queues with Gaussian input Mandjes (2007)

$$u\left(b\right) := P\left(\max_{k\geq 0} X\left(k\right) > b\right)$$

where $X\left(\cdot\right)$ is a suitably defined Gaussian process.

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- $\widetilde{P}$ is called a change-of-measure

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- Glasserman and Kou '95, Asmussen, Binswanger and Hojgaard '00

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$$\textit{Strong} \succeq \textit{Weak} \succeq \textit{Polynomial}$$

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- Suppose we choose $\widetilde{P}(\cdot) = P(\cdot | A)$

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- Estimator has *zero variance*, but requires knowledge of $P(A)$
- **Lesson:** Try choosing $\widetilde{P}(\cdot)$ close to $P(\cdot|\, A)$!

Outline

Example 1: Maximum of Gaussian Process

- Let $\mathbf{X} = (X_k : k \geq 0)$ be a Gaussian process with $Var(X_k) = \sigma_k^2$ and $EX_k = \mu_k < 0$.
- Want to efficiently estimate (as $b \nearrow \infty$)

$$u(b) = \mathbb{P}[\max_{k \geq 1} X_k > b]$$

- Assume

$$\sigma_k \sim c_\sigma k^{H_\sigma}, \quad |\mu_k| \sim c_\mu k^{H_\mu}, \quad 0 < H_\sigma < H_\mu < \infty$$

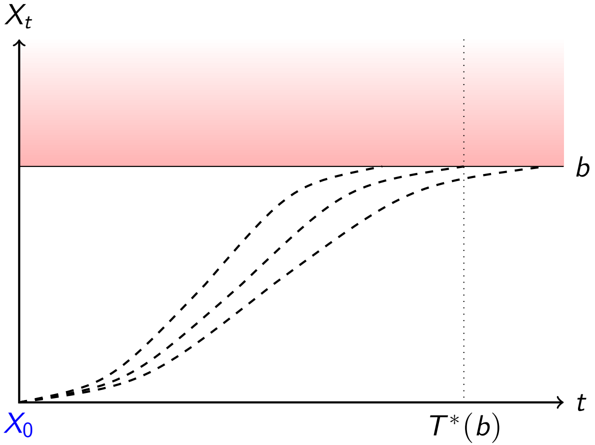
  so $u(b) \to 0$ as $b \to \infty$.
- No assumption on correlation structure.
- Asymptotic regime known as "large buffer scaling".

Related Asymptotics

- Rich literature on asymptotics for $u(b)$
  - Pickands (1969), Berman (1990), Duffield and O'Connell (1995), Piterbarg (1996), Husler and Piterbarg (1999), Dieker (2006), Husler (2006), Likhanov and Mazumdar (1999), Debicki and Mandjes (2003)...

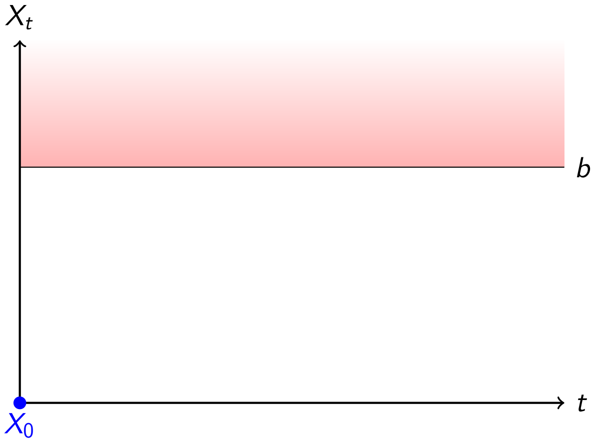
Related Simulation Algorithms

Common basic idea is to sample Gaussian process by mean tracking the most likely path given the overflow, time-slot by time-slot.



Target Bridge Sampling

1. Identifying the target set $\mathcal{T}$;



Target Bridge Sampling

1. Identifying the target set $\mathcal{T}$;
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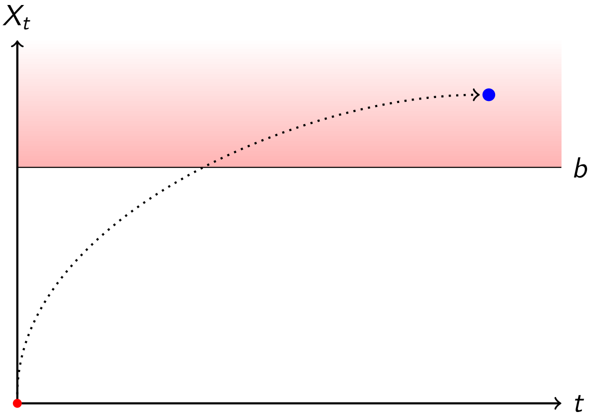

Target Bridge Sampling

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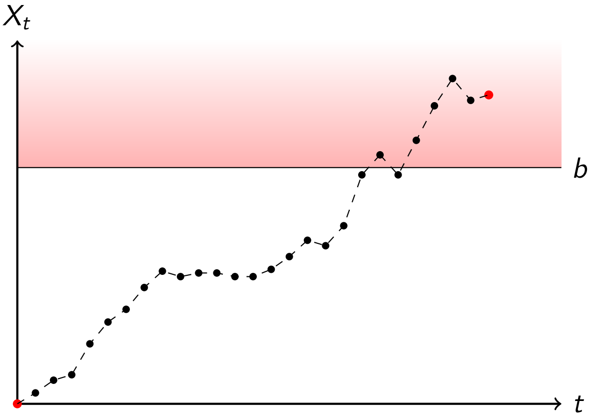

Target Bridge Sampling

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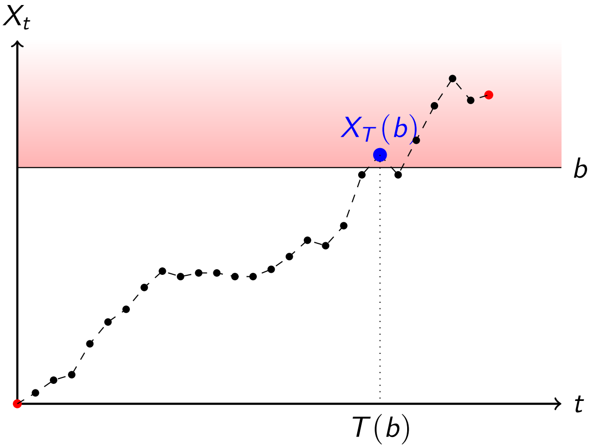
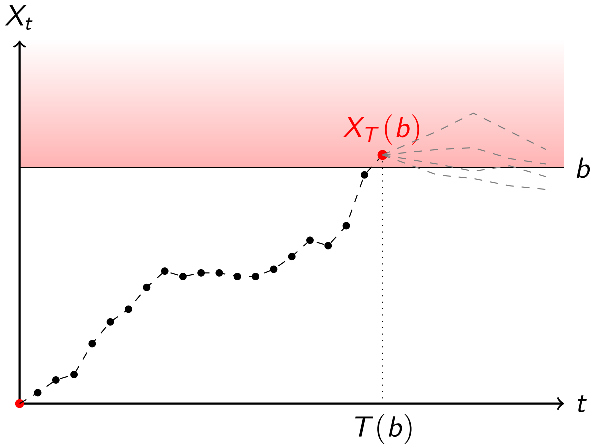
4. $T(b) = \inf\{k : X_k > b\}$;



Target Bridge Sampling

1. Identifying the target set $\mathcal{T}$;
2. Targeting;
3. Bridging;
4. $T(b) = \inf\{k : X_k > b\}$;
5. Deleting



Likelihood of a Path

$$
\begin{aligned}
&\widetilde{P}(X_1,\ldots,X_{T(b)}) \\
= \quad &\sum_{k=T(b)}^{\infty} \widetilde{P}\left(\tau = k\right) \int_b^{\infty} P(X_1,\ldots,X_{T(b)}|X_k)\widetilde{P}\left(dX_k\right).
\end{aligned}
$$

- Select

$$
\widetilde{P}\left(\tau = k\right) \propto P\left(X_k > b \middle| T\left(b\right) < \infty\right) \propto P\left(X_k > b\right)
$$

- Given $\tau = k$ sample $\widetilde{P}\left(X_k \in \cdot\right) = P\left(X_k \in \cdot \middle| X_k > b\right)$

Likelihood Ratio

$$
\begin{aligned}
& \widetilde{P}(X_1, \ldots, X_{T(b)}) \\
= & \sum_{k=T(b)}^{\infty} \widetilde{P}(\tau = k) \int_b^{\infty} P(X_1, \ldots, X_{T(b)} | X_k) \widetilde{P}(dX_k) \\
= & \sum_{k=T(b)}^{\infty} \frac{P(X_k > b)}{\sum_{j=1}^{\infty} P(X_j > b)} \int_b^{\infty} P(X_1, \ldots, X_{T(b)} | X_k) \frac{P(dX_k)}{P(X_k > b)} \\
= & \sum_{k=T(b)}^{\infty} \frac{P(X_1, \ldots, X_{T(b)}, X_k > b)}{\sum_{j=1}^{\infty} P(X_j > b)} \\
= & \frac{P(X_1, \ldots, X_{T(b)})}{\sum_{j=1}^{\infty} P(X_j > b)} \sum_{k=T(b)}^{\infty} P\left(X_k > b | X_1, \ldots, X_{T(b)}\right)
\end{aligned}
$$

Consequently, the importance sampling estimator for $u(b)$ generated by P is simply

$$
\begin{aligned}
L &= \frac{dP}{d\widetilde{P}}\left(X_1, ..., X_{T(b)}\right) \\
&= \frac{\sum_{j=1}^{\infty} P\left(X_j > b\right)}{\sum_{j=T(b)}^{\infty} P\left(X_j > b \middle| X_1, ..., X_{T(b)}\right)}.
\end{aligned}
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Special Case: Brownian Motion

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$$X_t = B(t) - t,$$

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- In this case TBS outputs a Brownian motion with drift $+1$.

Efficiency Analysis

- The efficiency analysis involves

$$\frac{\widetilde{E}(L^2)}{P(\max_{k\geq 1} X_k \geq b)^2} \leq \left[\frac{\sum_{k=0}^{\infty} P(X_k > b)}{P(\max_{k\geq 1} X_k \geq b)}\right]^2$$

$$\leq \left[\frac{\sum_{k=1}^{\infty} P(X_k \geq b)}{\max_{k\geq 1} P(X_k \geq b)}\right]^2$$

- Does **not involve** the correlation structure...

Efficiency Analysis

Theorem

If $(\sigma_k : k \geq 1)$ *and* $(\mu_k : k \geq 1)$ *have power law type tails with power indices* $0 < H_\sigma < H_\mu < \infty$ *respectively, then we have that*

1. L *is an unbiased estimator of* $u(b)$
2. *Let* $h(b) = \lambda\left(H_\mu, H_\sigma\right) b^{\frac{H_\mu - H_\sigma}{H_\mu}}$, *then*
$$u(b) = O(h(b)^{1/(H_\mu - H_\sigma)} \exp(-h(b))).$$
3. $$\frac{\widetilde{E}\left(L^2\right)}{u(b)^2} = O(b^{2/H_\mu});$$

- Polynomially efficient
- Strongly efficient in many source scaling

Summary of Performance Analysis (Updated)

Method	Many Sources	Large Buffer	Cost of each replication
Single twist	x	x	$O(b^3)$
Cut-and-twist	weakly	x	$O(b^4)$
Random twist	weakly	x	$O(b^3)$
Sequential twist	weakly	x	$O(nb^3)$
Mean shift	x	x	$O(b^3)$
BMC	x	x	$O(b^3)$
TBS	strongly	polynomial	$O(b^3)$

Numerical Example: Fractional Brownian Noise

- Test the performance of our Target Bridge Sampler and compare it against other existing methods in the many sources setting.
- Suppose that $\{X_k\}$ are driven by fractional Brownian noises, that is, $\mathrm{Cov}(X_k, X_j) = (k^{2H_\sigma} + j^{2H_\sigma} - |k-j|^{2H_\sigma})/2$ and $\mu_k = k$.
- The numerical result is compared against what was reported in Dieker and Mandjes (2006).

Numerical Example

$b = 300$	Cost of each replication	Estimator	Simulation Runs	Time
Naive	$O(b^3)$	6.12×10^{-4}	833562	232s
Single twist	$O(b^3)$	4.84×10^{-4}	4038	~60s
Cut-and-twist	$O(b^4)$	5.95×10^{-4}	703	~80s
Random twist	$O(b^3)$	5.50×10^{-4}	3269	~50s
Sequential twist	$O(nb^3)$	6.39×10^{-4}	692	~100s
TBS	$O(b^3)$	5.84×10^{-4}	26	1s
Benchmark	-	5.75×10^{-4}	-	

Table: Simulation result of Example 2 with $n = 300$, $b = 3$, $H_\sigma = 0.8$.

Outline

Discrete Version: Monte Carlo Strategy

- Discrete version: $(X_1, ..., X_d)$ multivariate Gaussian: Consider

$$P\left(\max_{i=1}^{d} X_i > b\right)$$

$$P\left((X_1, ..., X_d) \in \cdot \,|\, \max_{i=1}^{d} X_i > b\right)$$

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- Monte Carlo strategy:

1. Pick *i-th* coordinate with probability proportional to $P(X_i > b)$
2. Sample X_i given $X_i > b$
3. Sample the remaining values given X_i

$$\widetilde{P}((X_1, ..., X_d) \in \cdot) = \sum_{i=1}^{d} \frac{P(X_i > b)\, P((X_1, ..., X_d) \in \cdot | X_i > b)}{\sum_{j=1}^{d} P(X_j > b)}$$

Discrete Version: Importance Sampling Distribution

- Prob. measure

$$
\begin{aligned}
\widetilde{P}\left(\left(X_1, \ldots, X_d\right) \in \cdot\right) &= \sum_{i=1}^{d} \frac{P\left(X_i > b\right) P\left(\left(X_1, \ldots, X_d\right) \in \cdot \middle| X_i > b\right)}{\sum_{j=1}^{d} P\left(X_j > b\right)} \\
&= \sum_{i=1}^{d} \frac{P\left(\left(X_1, \ldots, X_d\right) \in \cdot ; X_i > b\right)}{\sum_{j=1}^{d} P\left(X_j > b\right)}
\end{aligned}
$$

Discrete Version: Likelihood Ratio

- Likelihood ratio

$$
\begin{aligned}
I(\max_{i=1}^{d} X_i &> b)\frac{dP}{d\widetilde{P}}(X_1, ..., X_d) \\
&= I(\max_{i=1}^{d} X_i > b)\frac{\sum_{j=1}^{d} P(X_j > b)}{\sum_{j=1}^{d} I(X_j > b)} \\
&\leq \sum_{j=1}^{d} P(X_j > b).
\end{aligned}
$$

Total Variation Approximation

Theorem (Adler, Blanchet and Liu (2008))

If $Corr(X_i, X_j) < 1$ *then*

$$P(\max_{i=1}^{d} X_i > b) = \sum_{j=1}^{d} P(X_j > b)(1 + o(1))$$

as $b \nearrow \infty$ *and therefore*

$$\sup_{A} \left| P((X_1, ..., X_d) \in A | \max_{i=1}^{d} X_i > b) - \widetilde{P}((X_1, ..., X_d) \in A) \right| \longrightarrow 0$$

as $b \nearrow \infty$.

Efficient Monte Carlo for High Excursions of Gaussian Random Fields

Jose Blanchet
Columbia University

Joint work with
Robert Adler (Technion-Israel Institute of Technology)
Jingchen Liu (Columbia University)

Continuous Gaussian Random Fields on Compacts

- Gaussian random field,

$$f(t,\omega) : T \times \Omega \to R$$

where $T \subset R^d$ is a compact set.

- $E(f(t)) = \mu(t)$

- $Cov(f(s), f(t)) = C(s,t), \qquad \sigma^2(t) = C(t,t)$

High Excursion Probability

- Interesting probability

$$P(\sup_{t\in T} f(t) > b)$$

as $b \to \infty$.

- More generally,

$$E\left(\Gamma\left(f\left(\cdot\right)\right) \middle| \sup_{t\in T} f\left(t\right) > b \right)$$

where $\Gamma(\cdot)$ is suitable functional.

Asymptotic results

- Under very mild conditions

$$\lim_{b\to\infty} \frac{\log P(\sup_{t\in T} f(t) > b)}{b^2} = -\frac{1}{\sup_{t\in T} 2\sigma^2(t)}$$

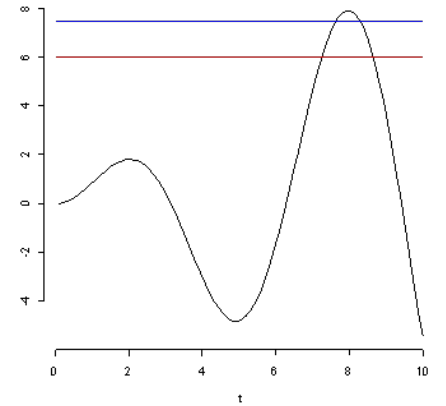
- Sharp asymptotics for mean zero and constant variance random fields (under conditions)

$$P(\sup_{t\in T} f(t) > b) = (1 + o(1))C(T) \times b^{k-1} \times e^{-\frac{b^2}{2\sigma^2}}$$

k depends on dimension of T and continuity of f(t).

- Pickands (1969), Piterbarg (1995), Sun (1993), Adler (1981), Azais and Wschebor (2005), Taylor, Takemura and Adler (2005).

The change of measure



- *mes* is the Lebesgue measure
- A_γ is the excursion set

$$A_\gamma = \{t \in T : f(t) > \gamma\}$$

$$E(mes(A_\gamma)) = E \int_T I(f(t) > \gamma)dt = \int_T P(f(t) > \gamma)dt$$

- A change of measure on C(T) $\quad \dfrac{dQ_\gamma}{dP} = \dfrac{mes(A_\gamma)}{E(mes(A_\gamma))}.$

Simulation from this change of measure

1. Simulate $\tau_\gamma \in T$

$$\tau_\gamma \sim h(t) = \frac{P(f(t) > \gamma)}{\int_T P(f(s) > \gamma) ds};$$

2. Simulate $f(\tau_\gamma)$ conditional on $f(\tau_\gamma) > \gamma$;

3. Simulate $\{f(t) : t \neq \tau_\gamma\}$ conditional on $f(\tau_\gamma)$.

The estimator and its variance

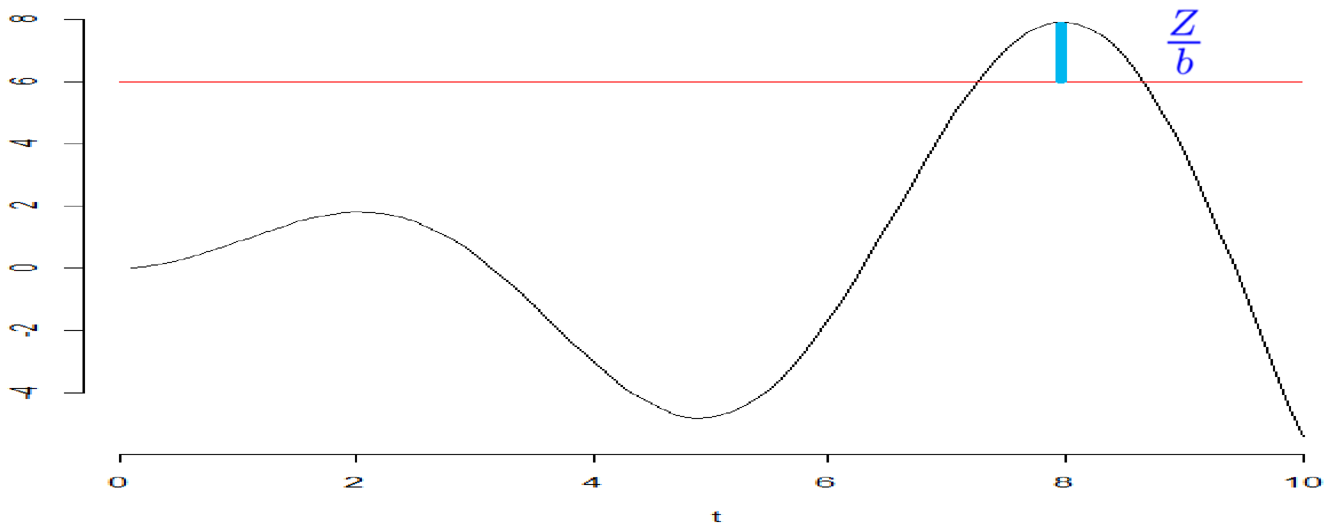
- The estimator

$$Z_b = I(\sup_T f(t) > b)\frac{\int_T P(f(t)>\gamma)dt}{mes(A_\gamma)}$$

- The second moment

$$\begin{aligned} E^Q Z_b^2 = E(Z_b) &= \int_T P(f(t) > \gamma)dt \times E\left(\frac{1}{mes(A_\gamma)}; \sup_T f(t) > b\right) \\ &= \int_T P(f(t) > \gamma)dt \\ &\times P\left(\sup_T f(t) > b\right) \\ &\times E\left(\frac{1}{mes(A_\gamma)}\middle| \sup_T f(t) > b\right) \end{aligned}$$

The Estimator and Its Variance



$$A_b = \{t : f(t) > b\},\ m(A_b) = \{t : f(t) > b\} \approx b^{-d} Z^{d/2}$$

$$Z \sim \text{Exponential}(1)$$

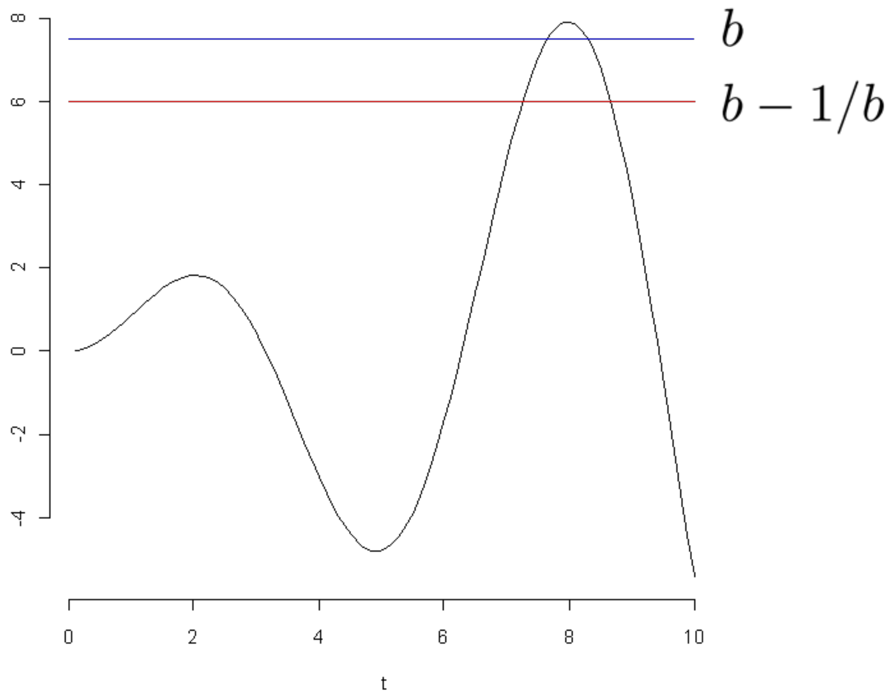
The choice of γ

- $\gamma = b$ results in infinite variance

- $\gamma = b - a/b$

$$
\begin{aligned}
E^Q Z_b^2 \quad = \quad & \int_T P(f(t) > b - a/b)dt \\
& \times P\left(\sup_T f(t) > b\right) \\
& \times E\left(\frac{1}{mes(A_{b-a/b})} \middle| \sup_T f(t) > b \right)
\end{aligned}
$$

The area of excursion set, $mes(A_{b-1/b})$, given high excursion



Efficiency results – the general case

Theorem 1 (Adler, Blanchet, and L. (2010)) *Choose* $\gamma = b - a/b$ *for some* $a > 0$ *and*

$$Z_b = I(\sup_T f(t) > b) \frac{\int_T P(f(t) > \gamma) dt}{mes(A_\gamma)}.$$

If f *is uniformly Hölder continuous, that is,* $E(f(s) - f(t))^2 \leq \kappa |s - t|^\beta$, *for some* $\beta \in (0, 2]$, *then*

$$\frac{E^Q Z_b^2}{P^2 (\sup_T f(t) > b)} \leq b^\alpha,$$

for some $\alpha > 0$ *and all* b.

Efficiency results – the homogeneous case

Theorem 1 (Adler, Blanchet, and L. (2010)) *Choose* $\gamma = b - a/b$ *for some* $a > 0$ *and*

$$Z_b = I(\sup_T f(t) > b) \frac{\int_T P(f(t) > \gamma) dt}{mes(A_\gamma)}.$$

If f *is twice differentiable and homogeneous, then*

$$E^Q Z_b^2 \le \kappa P^2 \left(\sup_T f(t) > b \right),$$

for some $\kappa > 0$ *and all* b.

Implementation – discretize the continuous filed

- Discretize the space T, $\{t_1,\ldots,t_n\}$

$$P\left(\sup_i f(t_i) > b\right) \to P\left(\sup_T f(t) > b\right), \text{ as } n \to \infty.$$

- It is sufficient to choose $n = (b/\varepsilon)^{\alpha}$ such that

$$1 - \varepsilon \le \frac{P(\sup_i f(t_i) > b)}{P(\sup_T f(t) > b)} \le 1$$

Adler, Blanchet and L. (2010)

Summary

- Non-exponential change-of-measure for Gaussian processes and fields (efficiency properties & conditional sampling)
- Polynomialy efficient for general Hölder continuous fields
- Strong efficiency for homogeneous fields