On Exact Sampling of Multidimensional SDEs

Jose Blanchet (joint work with X. Chen, J. Dong, K. Murthy)

Columbia University

May, 2014

Outline

Agenda

1. What do we know about exact sampling of SDEs?

Agenda

1. What do we know about exact sampling of SDEs?
2. Exact simulation of multidimensional RBM

Agenda

1. What do we know about exact sampling of SDEs?
2. Exact simulation of multidimensional RBM
3. SDEs with non-gradient drift vector fields

Outline

Acceptance / Rejection

- GOAL: Sample ω under $P(\cdot)$ <– HARD problem.

Acceptance / Rejection

- GOAL: Sample ω under $P(\cdot)$ <– HARD problem.
- Assume easier to sample ω under $Q(\cdot)$, and there is $c \in (0, \infty)$ *deterministic* so that

$$0 \leq p(\omega) := \frac{1}{c} \frac{dP}{dQ}(\omega) \leq 1.$$

Acceptance / Rejection

- GOAL: Sample ω under $P(\cdot)$ <– HARD problem.
- Assume easier to sample ω under $Q(\cdot)$, and there is $c \in (0, \infty)$ *deterministic* so that

$$0 \leq p(\omega) := \frac{1}{c} \frac{dP}{dQ}(\omega) \leq 1.$$

- **If** $B(\omega)$ **is Bernoulli**$(p(\omega))$ **under** $Q(\cdot)$**, then**

$$P(\omega \in \cdot) = Q(\omega \in \cdot \, | B(\omega) = 1).$$

Outline

Exact sampling of an SDE: One dimensional setting...

- Under $P(\cdot)$ the 1-dimensional process $X(\cdot)$ satisfies,

$$dX(t) = \mu'(X(t))\,dt + dB(t); \quad X(0) = x.$$

Exact sampling of an SDE: One dimensional setting...

- Under $P(\cdot)$ the 1-dimensional process $X(\cdot)$ satisfies,

$$dX(t) = \mu'(X(t))\,dt + dB(t)\,; \quad X(0) = x.$$

- **How to sample $X(T)$ with NO BIAS?**

Exact sampling of an SDE: One dimensional setting...

- Under $P(\cdot)$ the 1-dimensional process $X(\cdot)$ satisfies,

$$dX(t) = \mu'(X(t))\,dt + dB(t); \quad X(0) = x.$$

- **How to sample $X(T)$ with NO BIAS?**
- $X(\cdot)$ is Brownian motion under $Q(\cdot)$, so given $X(T)$

$$\frac{dP_x}{dQ_x}(X(T)) = e^{\mu(X(T))-\mu(x)} E_x^Q\left(e^{-\int_0^T \frac{\{\mu''(X(s))+\mu'(X(s))^2\}}{2}ds} | X(T)\right).$$

Exact sampling of an SDE: One dimensional setting...

- Under $P(\cdot)$ the 1-dimensional process $X(\cdot)$ satisfies,

$$dX(t) = \mu'(X(t))\,dt + dB(t); \quad X(0) = x.$$

- **How to sample $X(T)$ with NO BIAS?**
- $X(\cdot)$ is Brownian motion under $Q(\cdot)$, so given $X(T)$

$$\frac{dP_x}{dQ_x}(X(T)) = e^{\mu(X(T))-\mu(x)} E_x^Q \left(e^{-\int_0^T \frac{\{\mu''(X(s))+\mu'(X(s))^2\}}{2} ds} | X(T) \right).$$

- Assume $|\mu(\cdot)|, \mu'(\cdot)^2, |\mu''(\cdot)| \leq a < 0$, define

$$\lambda(X(s)) := \frac{\{\mu''(X(s)) + \mu'(X(s))^2\}}{2} + a \geq 0.$$

Exact sampling of an SDE: One dimensional setting...

- Under $P(\cdot)$ the 1-dimensional process $X(\cdot)$ satisfies,

$$dX(t) = \mu'(X(t))\,dt + dB(t); \quad X(0) = x.$$

- **How to sample $X(T)$ with NO BIAS?**
- $X(\cdot)$ is Brownian motion under $Q(\cdot)$, so given $X(T)$

$$\frac{dP_x}{dQ_x}(X(T)) = e^{\mu(X(T))-\mu(x)} E_x^Q \left(e^{-\int_0^T \frac{\{\mu''(X(s))+\mu'(X(s))^2\}}{2} ds} | X(T) \right).$$

- Assume $|\mu(\cdot)|, \mu'(\cdot)^2, |\mu''(\cdot)| \leq a < 0$, define

$$\lambda(X(s)) := \frac{\{\mu''(X(s)) + \mu'(X(s))^2\}}{2} + a \geq 0.$$

- REMEMBER $|\mu(\cdot)|, \mu'(\cdot)^2, |\mu''(\cdot)| \leq a < 0$ FOR NEXT SLIDE ONLY!

Exact sampling of an SDE: One dimensional setting...

- Apply Acceptance / Rejection, sample $X(T) \sim x + B^{Q}(T) =_d x + N(0, T)$,

$$\frac{dP_x}{dQ_x}(X(T)) = e^{\mu(X(T)) - \mu(x) - Ta} E_x^{Q}\left(e^{-\int_0^T \lambda(X(s))ds} | X(T)\right)$$
$$\leq e^{a - \mu(x) - Ta}.$$

Exact sampling of an SDE: One dimensional setting...

- Apply Acceptance / Rejection, sample $X(T) \sim x + B^Q(T) =_d x + N(0, T)$,

$$\frac{dP_x}{dQ_x}(X(T)) = e^{\mu(X(T)) - \mu(x) - Ta} E_x^Q \left(e^{-\int_0^T \lambda(X(s))ds} | X(T) \right)$$
$$\leq e^{a - \mu(x) - Ta}.$$

- Here $c = \exp(a - \mu(x) + Ta)$ and

$$p(X(T)) := \frac{1}{c} \frac{dP_x}{dQ_x}(X(T)) = \frac{e^{\mu(X(T))}}{e^a} \times E_x^Q \left(e^{-\int_0^T \lambda(X(s))ds} | X(T) \right)$$

Exact sampling of an SDE: One dimensional setting...

- Apply Acceptance / Rejection, sample $X(T) \sim x + B^{Q}(T) =_d x + N(0, T)$,

$$\frac{dP_x}{dQ_x}(X(T)) = e^{\mu(X(T)) - \mu(x) - Ta} E_x^Q \left(e^{-\int_0^T \lambda(X(s))ds} | X(T) \right) \leq e^{a - \mu(x) - Ta}.$$

- Here $c = \exp(a - \mu(x) + Ta)$ and

$$p(X(T)) := \frac{1}{c} \frac{dP_x}{dQ_x}(X(T)) = \frac{e^{\mu(X(T))}}{e^{a}} \times E_x^Q \left(e^{-\int_0^T \lambda(X(s))ds} | X(T) \right)$$

- **Accepting $X(T)$ reduces to checking if NO ARRIVALS occur in $[0, T]$ from a Cox process with intensity $\lambda(X(\cdot))$ where $X(\cdot)$ is Brownian bridge.. <- use thinning theorem.**

What we know about exact simulation of SDEs...

- Previous strategy due to Beskos & Roberts ('04).

What we know about exact simulation of SDEs...

- Previous strategy due to Beskos & Roberts ('04).
- Every single extension so far uses Poisson thinning reduction trick & ONLY in 1 dimension.

What we know about exact simulation of SDEs...

- Previous strategy due to Beskos & Roberts ('04).
- Every single extension so far uses Poisson thinning reduction trick & ONLY in 1 dimension.
- Extensions include: relaxing boundedness assumptions, jumps, reflection. Beskos, Roberts, & Papaspiliopoulos '06, Chen & Huang '12, Étoréa & Martinez (2011).

What we know about exact simulation of SDEs...

- Previous strategy due to Beskos & Roberts ('04).
- Every single extension so far uses Poisson thinning reduction trick & ONLY in 1 dimension.
- Extensions include: relaxing boundedness assumptions, jumps, reflection. Beskos, Roberts, & Papaspiliopoulos '06, Chen & Huang '12, Étoréa & Martinez (2011).
- SEVERE LIMITATIONS:

$$dX(t) = \nabla\mu(X(t))\,dt + dB(t)$$

What we know about exact simulation of SDEs...

- Previous strategy due to Beskos & Roberts ('04).
- Every single extension so far uses Poisson thinning reduction trick & ONLY in 1 dimension.
- Extensions include: relaxing boundedness assumptions, jumps, reflection. Beskos, Roberts, & Papaspiliopoulos '06, Chen & Huang '12, Étoréa & Martinez (2011).
- SEVERE LIMITATIONS:

$$dX(t) = \nabla \mu(X(t))\, dt + dB(t)$$

- **Drift needs to be a gradient & constant diffusion coefficient...**

What do we do?

Our contribution: Introduce a wide range of techniques enabling acceptance/rejection much more widely...

We illustrate in two settings: RBM and multidimensional diffusions of the form

$$dX(t) = \mu(X(t))\,dt + dB(t)$$

(i.e. drift may not be a gradient).

Outline

Multidimensional RBM: Harrison-Reiman '81 definition

- Given $R = I - Q'$, with Q substochasic and $Q^n \to \infty$

Multidimensional RBM: Harrison-Reiman '81 definition

- Given $R = I - Q'$, with Q substochasic and $Q^n \to \infty$
- $X(\cdot)$ is a standard Brownian motion in d-dimensions

Multidimensional RBM: Harrison-Reiman '81 definition

- Given $R = I - Q'$, with Q substochasic and $Q^n \to \infty$
- $X(\cdot)$ is a standard Brownian motion in d-dimensions
- *Skorokhod problem:* Find $(Y(\cdot), L(\cdot))$, a pair of process such that

$$dY(t) = dX(t) + RdL(t),$$
$$Y(\cdot) \geq 0, \quad Y_i(t)\, dL_i(t) = 0, \;\; dL_i(t) \geq 0.$$

Multidimensional RBM: Harrison-Reiman '81 definition

- Given $R = I - Q'$, with Q substochasic and $Q^n \to \infty$
- $X(\cdot)$ is a standard Brownian motion in d-dimensions
- *Skorokhod problem:* Find $(Y(\cdot), L(\cdot))$, a pair of process such that

$$\begin{aligned} dY(t) &= dX(t) + RdL(t), \\ Y(\cdot) &\geq 0, \quad Y_i(t)\, dL_i(t) = 0, \ dL_i(t) \geq 0. \end{aligned}$$

- $Y(\cdot)$ **is called multidimensional RBM.**

Exact Simulation of RBM: Our contribution here

Theorem (B. and Murthy '14)

One can sample exactly $Y(T)$ for a multidimensional RBM in finite time.

- **Remark 1:** Methodology extends easily to multidimensional reflected diffusions of the form

$$dY(t) = \nabla u(Y(t))\,dt + dB(t) + dL(t), \quad Y(0) = y_0.$$

Our Strategy in a Nutshell: Forget RBM for a moment...

- Consider $Y = W + \Delta$ where $W \in [2\delta, 3\delta]$ and $\Delta \in [-\delta, \delta]$ are independent

Our Strategy in a Nutshell: Forget RBM for a moment...

- Consider $Y = W + \Delta$ where $W \in [2\delta, 3\delta]$ and $\Delta \in [-\delta, \delta]$ are independent
- Density $f_{\Delta}(\cdot)$ is bounded by C' & Lipschitz continuous

Our Strategy in a Nutshell: Forget RBM for a moment...

- Consider $Y = W + \Delta$ where $W \in [2\delta, 3\delta]$ and $\Delta \in [-\delta, \delta]$ are independent
- Density $f_{\Delta}(\cdot)$ is bounded by C' & Lipschitz continuous
- Given, $W = w$, density of Y is $f_Y(z|w) = f_{\Delta}(z - w)$

Our Strategy in a Nutshell: Forget RBM for a moment...

- Consider $Y = W + \Delta$ where $W \in [2\delta, 3\delta]$ and $\Delta \in [-\delta, \delta]$ are independent
- Density $f_\Delta(\cdot)$ is bounded by C' & Lipschitz continuous
- Given, $W = w$, density of Y is $f_Y(z|w) = f_\Delta(z - w)$
- TO SAMPLE Y: Sample W & propose $Z \sim U(\delta, 4\delta)$, get likelihood ratio

$$\frac{f_Y(Z|W)}{1/3\delta} = 3\delta f_\Delta(Z - W) \le c := 3\delta C'$$

Our Strategy in a Nutshell: Forget RBM for a moment...

- Consider $Y = W + \Delta$ where $W \in [2\delta, 3\delta]$ and $\Delta \in [-\delta, \delta]$ are independent
- Density $f_\Delta(\cdot)$ is bounded by C' & Lipschitz continuous
- Given, $W = w$, density of Y is $f_Y(z|w) = f_\Delta(z - w)$
- TO SAMPLE Y: Sample W & propose $Z \sim U(\delta, 4\delta)$, get likelihood ratio

$$\frac{f_Y(Z|W)}{1/3\delta} = 3\delta f_\Delta(Z - W) \leq c := 3\delta C'$$

- Let $V \sim U(0, 1)$ independent of everything and accept Z IF

$$V \leq 3\delta f_\Delta(Z - W)/c = f_\Delta(Z - W)/C'.$$

Our Strategy in a Nutshell: Forget RBM for a moment...

- Consider $Y = W + \Delta$ where $W \in [2\delta, 3\delta]$ and $\Delta \in [-\delta, \delta]$ are independent
- Density $f_\Delta(\cdot)$ is bounded by C' & Lipschitz continuous
- Given, $W = w$, density of Y is $f_Y(z|w) = f_\Delta(z - w)$
- TO SAMPLE Y: Sample W & propose $Z \sim U(\delta, 4\delta)$, get likelihood ratio

$$\frac{f_Y(Z|W)}{1/3\delta} = 3\delta f_\Delta(Z - W) \leq c := 3\delta C'$$

- Let $V \sim U(0, 1)$ independent of everything and accept Z IF

$$V \leq 3\delta f_\Delta(Z - W)/c = f_\Delta(Z - W)/C'.$$

- **Key observation: Don't need to know W! Suffices to have $|W_n - W| \leq \varepsilon_n \to 0$! Sample FIRST Z and if**

$$V < \frac{f_\Delta(Z - W_n)}{C'} - K\varepsilon_n \quad \text{—-> ACCEPT}$$

$$V > \frac{f_\Delta(Z - W_n)}{C'} + K\varepsilon_n \quad \text{—-> REJECT}$$

Exact Simulation of RBM: Use Following Facts

- **FACT 1:** $Y(\cdot\,; X)$ is Lipschitz in $X(\cdot)$. That is for $K > 0$ computable

$$\max_{t\in[0,1]} \left| Y(t; X) - Y(t; X') \right| \leq K \max_{t\in[0,1]} \left| X(t) - X'(t) \right|.$$

Exact Simulation of RBM: Use Following Facts

- **FACT 1:** $Y(\cdot\ ; X)$ is Lipschitz in $X(\cdot)$. That is for $K > 0$ computable

$$\max_{t\in[0,1]} \left| Y(t; X) - Y(t; X') \right| \leq K \max_{t\in[0,1]} \left| X(t) - X'(t) \right|.$$

- **FACT 2:** $P(Y(t) > 0) = 1$ (deterministic t) and $Y(\cdot)$ is continuous.

Exact Simulation of RBM: Use Following Facts

- **FACT 1:** $Y(\cdot\,; X)$ is Lipschitz in $X(\cdot)$. That is for $K > 0$ computable

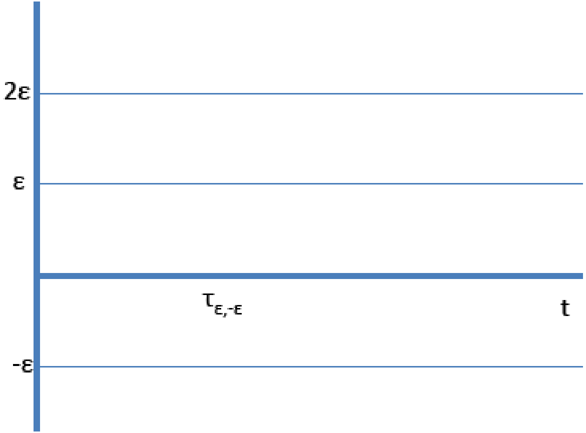
$$\max_{t\in[0,1]} \left| Y(t; X) - Y(t; X') \right| \leq K \max_{t\in[0,1]} \left| X(t) - X'(t) \right|.$$

- **FACT 2:** $P(Y(t) > 0) = 1$ (deterministic t) and $Y(\cdot)$ is continuous.
- **FACT 3** (Beskos, Peluchetti, Roberts '12 & B. Chen '13): Can simulate $X_\varepsilon(\cdot)$ piecewise linear such that **with probability one**

$$\max_{t\in[0,1]} \left| X(t) - X_\varepsilon(t) \right| < \varepsilon.$$

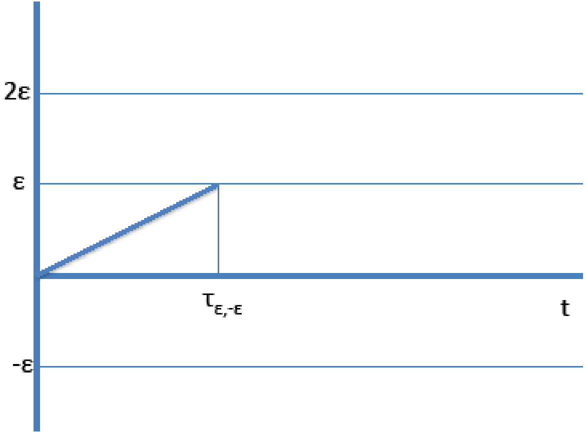
Exact Simulation of RBM: Using uniform simulation approximations

- **Simulate by stopping times...**



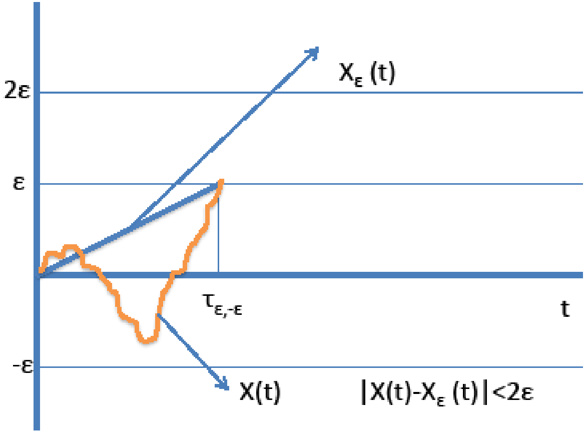
Exact Simulation of RBM: Using uniform simulation approximations

- **Simulate by stopping times...**



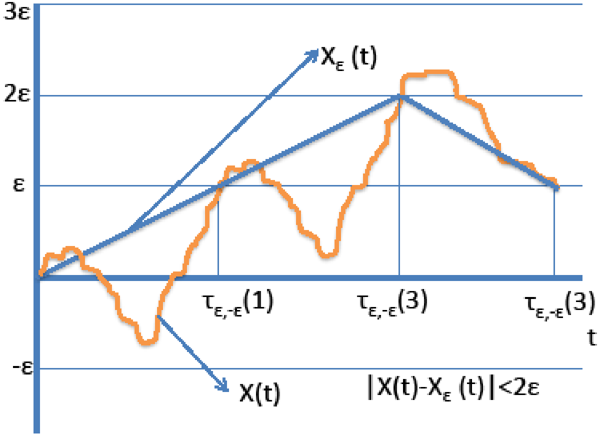
Exact Simulation of RBM: Using uniform simulation approximations

- **Simulate by stopping times...**



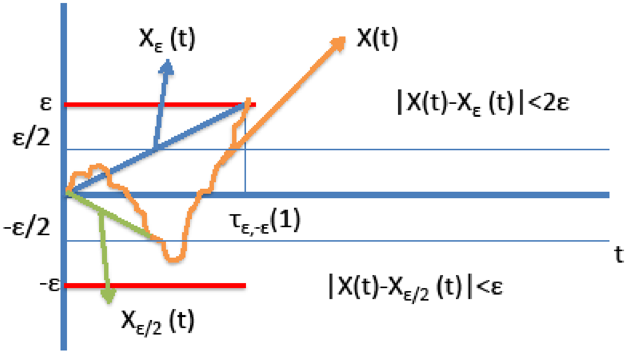
Exact Simulation of RBM: Using uniform simulation approximations

- **Simulate by stopping times...**



Exact Simulation of RBM: Using uniform simulation approximations

- **Refining $\varepsilon/2$: Sampling from conditional BESSEL BRIDGE $<$— Known transition density!**



Exact Simulation of RBM: Algorithm

- Simulate $X_{\varepsilon_1}(\cdot)$, $X_{\varepsilon_2}(\cdot)$,...,$X_{\varepsilon_N}(\cdot)$, $\varepsilon_N = 2^{-N}$ until $Y_{\varepsilon_N}(s) > 0$ for all $s \in [\tau_-, \tau_+]$ & $T \in [\tau_-, \tau_+]$.

Exact Simulation of RBM: Algorithm

- Simulate $X_{\varepsilon_1}(\cdot)$, $X_{\varepsilon_2}(\cdot)$,...,$X_{\varepsilon_N}(\cdot)$, $\varepsilon_N = 2^{-N}$ until $Y_{\varepsilon_N}(s) > 0$ for all $s \in [\tau_-, \tau_+]$ & $T \in [\tau_-, \tau_+]$.
- Stop in finite time: By FACT 2, $Y(T) > 0$ almost surely & $Y(\cdot)$ continuous.

Exact Simulation of RBM: Algorithm

- Simulate $X_{\varepsilon_1}(\cdot)$, $X_{\varepsilon_2}(\cdot)$,...,$X_{\varepsilon_N}(\cdot)$, $\varepsilon_N = 2^{-N}$ until $Y_{\varepsilon_N}(s) > 0$ for all $s \in [\tau_-, \tau_+]$ & $T \in [\tau_-, \tau_+]$.
- Stop in finite time: By FACT 2, $Y(T) > 0$ almost surely & $Y(\cdot)$ continuous.
- Let $\mathcal{F}_N(\tau_-)$ = information generated by $\{X_N(s) : s \leq \tau_-\}$.

Exact Simulation of RBM: Algorithm

- Simulate $X_{\varepsilon_1}(\cdot)$, $X_{\varepsilon_2}(\cdot)$,...,$X_{\varepsilon_N}(\cdot)$, $\varepsilon_N = 2^{-N}$ until $Y_{\varepsilon_N}(s) > 0$ for all $s \in [\tau_-, \tau_+]$ & $T \in [\tau_-, \tau_+]$.
- Stop in finite time: By FACT 2, $Y(T) > 0$ almost surely & $Y(\cdot)$ continuous.
- Let $\mathcal{F}_N(\tau_-) =$ information generated by $\{X_N(s) : s \leq \tau_-\}$.
- Note that

$$Y(T) = Y(\tau_-) + X(T) - X(\tau_-) \quad \text{AND}$$

Exact Simulation of RBM: Algorithm

- Simulate $X_{\varepsilon_1}(\cdot)$, $X_{\varepsilon_2}(\cdot)$,...,$X_{\varepsilon_N}(\cdot)$, $\varepsilon_N = 2^{-N}$ until $Y_{\varepsilon_N}(s) > 0$ for all $s \in [\tau_-, \tau_+]$ & $T \in [\tau_-, \tau_+]$.
- Stop in finite time: By FACT 2, $Y(T) > 0$ almost surely & $Y(\cdot)$ continuous.
- Let $\mathcal{F}_N(\tau_-) =$ information generated by $\{X_N(s) : s \leq \tau_-\}$.
- Note that

$$Y(T) = Y(\tau_-) + X(T) - X(\tau_-) \quad \text{AND}$$

- $\Delta = X(T) - X(\tau_-)$ is increment of conditional Bessel bridge so KNOWN density $f_\Delta(\cdot)$

Exact Simulation of RBM: Algorithm

- Simulate $X_{\varepsilon_1}(\cdot)$, $X_{\varepsilon_2}(\cdot)$,...,$X_{\varepsilon_N}(\cdot)$, $\varepsilon_N = 2^{-N}$ until $Y_{\varepsilon_N}(s) > 0$ for all $s \in [\tau_-, \tau_+]$ & $T \in [\tau_-, \tau_+]$.
- Stop in finite time: By FACT 2, $Y(T) > 0$ almost surely & $Y(\cdot)$ continuous.
- Let $\mathcal{F}_N(\tau_-)$ = information generated by $\{X_N(s) : s \leq \tau_-\}$.
- Note that

$$Y(T) = Y(\tau_-) + X(T) - X(\tau_-) \quad \text{AND}$$

- $\Delta = X(T) - X(\tau_-)$ is increment of conditional Bessel bridge so KNOWN density $f_\Delta(\cdot)$
- **RESULT:** $f_\Delta(\cdot)$ **is Lipschitz continuous with support inside** $[-2^{-N+1}, 2^{-N+1}]$**.**

Exact Simulation of RBM: Algorithm

- Apply acceptance rejection: Let $f_{Y(T)}(\cdot)$ be density of $Y(T)$ given $\mathcal{F}_{N_0}(\tau_-)$

$$f_{Y(T)}(z) = f_{\Delta}(z - Y(\tau_-)).$$

Exact Simulation of RBM: Algorithm

- Apply acceptance rejection: Let $f_{Y(T)}(\cdot)$ be density of $Y(T)$ given $\mathcal{F}_{N_0}(\tau_-)$
$$f_{Y(T)}(z) = f_{\Delta}(z - Y(\tau_-)).$$
- We know that $Y(T) \in [Y_{\varepsilon_N}(\tau_-) - K2^{-N}, Y_{\varepsilon_N}(\tau_-) + K2^{-N}]$ for computable K (FACT 1: Lipschitz continuity of Skorokhod map)

Exact Simulation of RBM: Algorithm

- Apply acceptance rejection: Let $f_{Y(T)}(\cdot)$ be density of $Y(T)$ given $\mathcal{F}_{N_0}(\tau_-)$
$$f_{Y(T)}(z) = f_\Delta(z - Y(\tau_-)).$$
- We know that $Y(T) \in [Y_{\varepsilon_N}(\tau_-) - K2^{-N}, Y_{\varepsilon_N}(\tau_-) + K2^{-N}]$ for computable K (FACT 1: Lipschitz continuity of Skorokhod map)
- Propose Z from uniformly on $[Y_{\varepsilon_N}(\tau_-) - K2^{-N}, Y_{\varepsilon_N}(\tau_-) + K2^{-N}]$ then accept Z as a sample from $f_{Y(T)}(\cdot)$ IF
$$V \leq \frac{1}{C(N)} f_\Delta(Z - Y(\tau_-)),$$
where V is $U(0,1)$ independent of everything.

Exact Simulation of RBM: Algorithm

- Apply acceptance rejection: Let $f_{Y(T)}(\cdot)$ be density of $Y(T)$ given $\mathcal{F}_{N_0}(\tau_-)$
$$f_{Y(T)}(z) = f_\Delta (z - Y(\tau_-)).$$
- We know that $Y(T) \in [Y_{\varepsilon_N}(\tau_-) - K2^{-N}, Y_{\varepsilon_N}(\tau_-) + K2^{-N}]$ for computable K (FACT 1: Lipschitz continuity of Skorokhod map)
- Propose Z from uniformly on $[Y_{\varepsilon_N}(\tau_-) - K2^{-N}, Y_{\varepsilon_N}(\tau_-) + K2^{-N}]$ then accept Z as a sample from $f_{Y(T)}(\cdot)$ IF
$$V \leq \frac{1}{C(N)} f_\Delta (Z - Y(\tau_-)),$$
where V is $U(0,1)$ independent of everything.
- BIG problem $Y(\tau_-)$ is unknown... is it really?

Exact Simulation of RBM: Algorithm

- Key observations:

$$Law\left(\Delta|\sigma(\cup_{k=N}^{\infty}\mathcal{F}_k\left(\tau_-\right))\right) = Law\left(\Delta|\mathcal{F}_N\left(\tau_-\right)\right)$$

  and missing information to finally evaluate $Y\left(\tau_-\right)$ is inside $\sigma(\cup_{k>N}^{\infty}\mathcal{F}_k\left(\tau_-\right))$.

Exact Simulation of RBM: Algorithm

- Key observations:

$$Law\left(\Delta|\sigma(\cup_{k=N}^{\infty}\mathcal{F}_k\left(\tau_-\right))\right) = Law\left(\Delta|\mathcal{F}_N\left(\tau_-\right)\right)$$

  and missing information to finally evaluate $Y\left(\tau_-\right)$ is inside $\sigma(\cup_{k>N}^{\infty}\mathcal{F}_k\left(\tau_-\right))$.

- So, you can continue refining $X_{\varepsilon_{N+1}}, X_{\varepsilon_{N+2}}, \ldots$to get $Y_{\varepsilon_{N+1}}\left(\tau_-\right)$, $Y_{\varepsilon_{N+2}}\left(\tau_-\right)$, $Y_{\varepsilon_{N+3}}\left(\tau_-\right)\ldots$ using Lipschitz continuity of $f_\Delta\left(\cdot\right)$ eventually

$$V \leq \frac{1}{C\left(N\right)}f_\Delta\left(Z - Y_{\varepsilon_{N+m}}\left(\tau_-\right)\right) - \frac{\widetilde{K}}{C\left(N\right)}\varepsilon_{N+m} \text{ -> ACCEPT}$$

OR

$$V \geq \frac{1}{C\left(N\right)}f_\Delta\left(Z - Y_{\varepsilon_{N+m}}\left(\tau_-\right)\right) + \frac{\widetilde{K}}{C\left(N\right)}\varepsilon_{N+m} \text{ -> REJECT}$$

Exact Simulation of RBM: Algorithm

- Key observations:

$$Law\left(\Delta|\sigma(\cup_{k=N}^{\infty}\mathcal{F}_k\left(\tau_-\right))\right) = Law\left(\Delta|\mathcal{F}_N\left(\tau_-\right)\right)$$

  and missing information to finally evaluate $Y\left(\tau_-\right)$ is inside $\sigma(\cup_{k>N}^{\infty}\mathcal{F}_k\left(\tau_-\right))$.

- So, you can continue refining $X_{\varepsilon_{N+1}}, X_{\varepsilon_{N+2}}, \ldots$to get $Y_{\varepsilon_{N+1}}\left(\tau_-\right)$, $Y_{\varepsilon_{N+2}}\left(\tau_-\right)$, $Y_{\varepsilon_{N+3}}\left(\tau_-\right)$... using Lipschitz continuity of $f_\Delta\left(\cdot\right)$ eventually

$$\begin{aligned} V &\leq \frac{1}{C\left(N\right)}f_\Delta\left(Z - Y_{\varepsilon_{N+m}}\left(\tau_-\right)\right) - \frac{\widetilde{K}}{C\left(N\right)}\varepsilon_{N+m} \text{ –> ACCEPT} \\ &\text{OR} \\ V &\geq \frac{1}{C\left(N\right)}f_\Delta\left(Z - Y_{\varepsilon_{N+m}}\left(\tau_-\right)\right) + \frac{\widetilde{K}}{C\left(N\right)}\varepsilon_{N+m} \text{ –> REJECT} \end{aligned}$$

- **Since $\varepsilon_n \to \infty$, algorithm must finish in finite time...**

Outline

The goal and the challenge

- GOAL: Sample without bias $X(1)$ such that

$$dX(t) = \mu(X(t))\,dt + dB(t)\,; \quad X(0) = x.$$

The goal and the challenge

- GOAL: Sample without bias $X(1)$ such that

$$dX(t) = \mu(X(t))\,dt + dB(t); \quad X(0) = x.$$

- CHALLENGE: How to "bound stoch. integral" in likelihood ratio?

$$L = \exp\left(\int_0^T \mu(X(s))\,dX(s) - \int_0^T \frac{\|\mu(X(s))\|^2}{2} ds\right)$$

Key tool strong simulation

Theorem (B., Chen, Dong '14)

Given $\mu(\cdot)$ *and* $\sigma(\cdot)$ *twice differentiable and Lipschitz*

$$\begin{aligned} dY(t) &= \mu(Y(t))\,dt + \sigma(Y(t))\,dB(t) \\ Y(0) &= x(0). \end{aligned}$$

We can construct $\{X_n(\cdot)\}$ *piecewise linear and jointly simulatable in a computer such that*

$$\sup_{t\in[0,1]} |Y_n(t) - Y(t)| < 1/n$$

with probability 1.

Apply strong simulation to decide if accept or reject

- CHALLENGE: How to "bound stoch. integral" in likelihood ratio?

$$L = \exp\left(\int_0^T \mu\left(X\left(s\right)\right) dX\left(s\right) - \int_0^T \frac{\left\|\mu\left(X\left(s\right)\right)\right\|^2}{2} ds\right)$$

Apply strong simulation to decide if accept or reject

- CHALLENGE: How to "bound stoch. integral" in likelihood ratio?

$$L = \exp\left(\int_0^T \mu\left(X\left(s\right)\right) dX\left(s\right) - \int_0^T \frac{\left\|\mu\left(X\left(s\right)\right)\right\|^2}{2} ds\right)$$

- Define

$$\begin{aligned} dX\left(t\right) &= \mu\left(X\left(t\right)\right) dt + dB\left(t\right) \\ dY\left(t\right) &= \left\|\mu\left(X\left(t\right)\right)\right\|_2^2 + \mu\left(X\left(t\right)\right) dB\left(t\right) \end{aligned}$$

and use strong simulation to approximate L to decide if accept or reject.

Outline

Key references behind our contributions

- Exact simulation of RBM: **http://arxiv.org/pdf/1405.6469v1.pdf**

Key references behind our contributions

- Exact simulation of RBM: **http://arxiv.org/pdf/1405.6469v1.pdf**
- ε-strong simulation of SDEs: **http://arxiv-web3.library.cornell.edu/abs/1403.5722v1**